Multigrid Solution of the Steady Euler Equations

S.P. Spekreijse
National Aerospace Laboratory, NLR
P.O. Box 90502, 1006 BM Amsterdam

This article gives a very short review of CWI Tract 46. This tract reflects the Ph.D.-work done by the author during the period 1983-1987 at CWI, at the Department of Numerical Mathematics.

Consider the flow field over an arbitrary aerodynamic body, for instance the wing of an aircraft. We are interested in calculating the properties of the flow field at all points in the flow. These properties are described by the pressure (p), the density (ρ), and the velocity ($\mathbf{v}=(v_1,v_2,v_3)$) in the field. The reason for calculating the flow properties throughout the flow is that it allows us to compute (among other things) the pressure distribution on the body, and thus the aerodynamic forces (lift and drag) and moments of the body (useful for an aircraft). For complex configurations, computational fluid dynamics (CFD) is the only tool to compute such flow fields.

In [3] it is assumed that the fluid is an inviscid, non-heat-conducting, perfect gas, without body forces. Then, in two dimensions, the principles of mass and energy conservation and Newton's second law result in the Euler equations, a hyperbolic system of nonlinear conservation laws

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial}{\partial x} f(\mathbf{q}) + \frac{\partial}{\partial y} g(\mathbf{q}) = 0,$$

where (x,y) are the Cartesian coordinates, t denotes the time, $\mathbf{q}=(\rho,\rho v_1,\rho v_2,E)^T$, a function of t, x and y, is the unknown state vector of conservative variables. For the perfect gas, E, the energy per unit volume, is related to the variables p,ρ and $\mathbf{v}$ as $E=p/(\gamma-1)+\rho(v_1^2+v_2^2)/2$, where $\gamma=1.4$ is the ratio of specific heats. The vector functions f and g are given by

$$f(\mathbf{q}) = (\rho v_1,\, \rho v_1^2+p,\, \rho v_1 v_2,\, (E+p)v_1)^T, \quad \text{and}$$
$$g(\mathbf{q}) = (\rho v_2,\, \rho v_1 v_2,\, \rho v_2^2+p,\, (E+p)v_2)^T.$$

To make the computation for a steady (i.e. time-independent) flow, the Euler equations for the continuous unknown functions, are discretised (i.e. written as a large system of nonlinear *algebraic* equations) in the following way. For the specific problem of interest, e.g. flow in a channel, physical space is subdivided in a finite number of disjoint quadrilateral 'control volumes' (see Figure 1) and the numerical approximations of the unknown variables are represented by their mean values over these small volumes. The flux at a control volume boundary represents the mass, momentum and energy transported per unit of time across that boundary. The discrete equations are obtained by the requirement that the total flux is zero for each such control volume. Thus, we obtain a nonlinear system to solve, in which there are 4 unknown quantities per control volume. A space discretization by N control volumes results in 4N unknown quantities and 4N algebraic equations.

A simple way to solve the large system of nonlinear algebraic equations is the Gauss-Seidel relaxation method. In this method all volumes are scanned one by one in some prescribed order and at each control volume visited the 4 unknown quantities are changed simultaneously by solving the 4 corresponding nonlinear equations by the Newton-Raphson method (local linearization). This iteration process is repeated until the solution of the large system has been obtained at a certain prescribed accuracy level. A disadvantage of the Gauss-Seidel method is that many iterations ($O(N^2)$) are necessary. Because of the local nature of the Gauss-Seidel method (the equations are only solved locally) short wavelength error components (with respect to the meshwidth), present in a current iterand, are damped efficiently but the damping of error components with a long wavelength is only marginal. For a grid with a typical meshwidth h the wavelength of a short wave is $O(h)$ and of a long wave is $O(1/h)$.

To accelerate the convergence of the Gauss-Seidel method, coarser grids (i.e. subdivisions in control volumes) are constructed recursively (see Figure 1). A coarser grid is obtained from a finer grid by assembling 4 fine grid control volumes into a single coarser grid volume. The discrete equations for the coarser grids are obtained in the same way as for the finest grid.

The reason for constructing coarser grids stems from the observation that a long wavelength error component (with respect to the meshwidth of the finest grid) is a short wavelength error component with respect to the meshwidth of a sufficiently coarser grid; the typical meshwidth of a coarse grid is two times the typical meshwidth of the next finer grid. As a consequence, on a certain coarse grid, the Gauss-Seidel relaxation method damps efficiently those long wavelength error components (with respect to the meshwidth of the finest grid) of which the wavelength is short with respect to the meshwidth of that particular coarse grid. This is the Multi-Grid idea which has been established in the seventies by the pioneering work of A. Brandt and others [1,2]. Hence, in the multigrid method, relaxations are not only performed on the finest grid but also on the coarser grids in order to dampen *all* error components efficiently. In the multigrid method, operators are necessary to exchange information between the grids at different levels. These operators are called prolongation operators (from a coarse to a next finer grid) and restriction operators (from a fine to a preceding coarser grid).

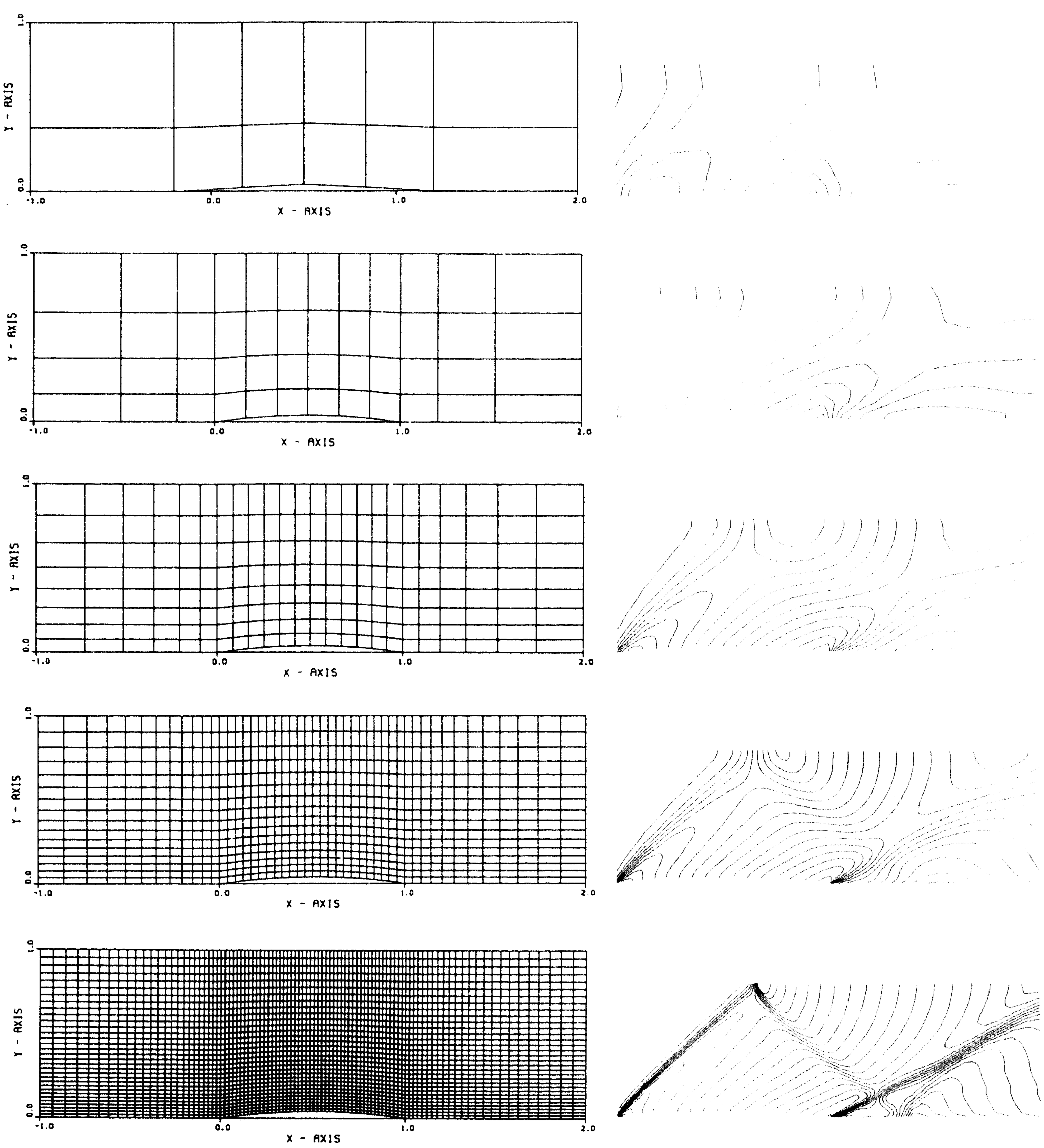

FIGURE 1. Illustration of the multi-grid method: the nested sequence of grids and the corresponding solutions (pressure distributions of a flow field) obtained during the multi-grid procedure.

The main advantage of multigrid over other acceleration techniques is the fact that the rate of convergence is independent of N, the size of the system to be solved. In fact, only a few iterations are sufficient to obtain a solution on the finest grid up to truncation error accuracy; due to discretization errors it is senseless to converge below truncation error accuracy. Another advantage of the multigrid method is that it can be used as a continuation process to obtain a good initial estimation of the solution on the finest grid; start the Gauss-Seidel relaxation on the coarsest grid, prolongate the solution to the next finer grid and apply the multigrid method on that grid. Repeat this process until the finest grid is reached. This continuation process is called nested iteration or the full multigrid method (FMG).

Figure 1 shows the solution on the several grids obtained during the full multigrid method. The flow is a supersonic flow (the Mach number at the inlet is 1.4) in a channel with a 4% thick circular arc bump. We see from the pressure distributions that strong oblique shocks are present in the flow field. A solution on a certain grid is depicted just before prolongation to a next finer grid. The solution on the finest grid is obtained with an amount of work (CPU time) equivalent to about 10 Gauss-Seidel relaxations on the finest grid. So far, such an efficiency in solving the steady Euler equations has never been obtained before.

References

1. A. Brandt (1982). Guide to multigrid development. W. Hackbusch, U. Trottenberg (eds.). *Multigrid Methods,* Lecture Notes in Mathematics 960, Springer Verlag, Berlin, 220-312.
2. W. Hackbusch (1985). *Multi-Grid Methods and Applications,* Springer Series in Computational Mathematics 4, Springer Verlag, Berlin.
3. S.P. Spekreijse (1988). *Multigrid Solution of the Steady Euler Equations,* CWI Tract 46, Centre for Mathematics and Computer Science, Amsterdam.